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THE SOLUTIONS OF PROBLEMS IN THE MATHEMATICAL THEORY OF PLASTICITY WITH DISCONTINUITIES IN THE DISPLACEMENT FIELDS†

S. I. REPIN

St Petersburg

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The properties of variational formulations of problems in the deformation theory of plasticity are investigated, in which the yield function depends on the first and second invariants of the stress tensor. Exact solutions containing discontinuities are constructed for some problems. It is shown that solutions containing discontinuities in the displacement field along certain lines (surfaces) can exist only in the case when the yield surface lies within a specified set.

As is well known, the yield function $F(\sigma)$ governing the plasticity properties of a material is usually assumed to be a convex function of the three invariants of the stress tensor σ. Below we consider variational formulations for displacement fields in plastic deformation problems in which $F(\sigma)$ depends on the first and second invariants of σ. In this case the yield condition can be written in the form

$$F(\mathrm{Sp}\sigma, |\sigma^D|) \leqslant 0 \tag{0.1}$$

where $\mathrm{Sp}\sigma$ is the first invariant of the tensor σ and σ^D is the deviator. Special cases of (0.1) are

$$F(\sigma) = |\sigma^D| - b \leqslant 0 \tag{0.2}$$

$$F(\sigma) = |\sigma^D| + a\mathrm{Sp}\sigma - b \leqslant 0 \tag{0.3}$$

$$F(\sigma) = |\sigma^D| + h(\mathrm{Sp}\sigma) - b \leqslant 0 \tag{0.4}$$

$$F(\sigma) = a^2(\mathrm{Sp}\sigma)^2 + |\sigma^D|^2 - b^2 \leqslant 0 \tag{0.5}$$

Here a and b are positive constants and h is a convex function. Condition (0.2) with $b = \sqrt{(2)}k_*$ (where k_* is the yield limit) corresponds to the von Mises yield criterion and is generally used for metals. Condition (0.3) is called the generalized Coulomb–Mohr criterion [1, 2] and is used to analyse the behaviour of soil, dry substances and granular media. The yield condition (0.4) is an extension of (0.3) [2], while (0.5) holds for porous bodies [3] and for problems of the plane stress states of thin plates [4]. In Fig. 1 the numbers 1–4 indicate the boundaries of domains $F(\sigma) \geqslant 0$ corresponding to conditions (0.2)–(0.5).

We know that discontinuous solutions can occur in ideal plasticity problems. Various researchers have investigated the conditions that arise along the lines of discontinuity (see, in particular, [4–7]). An example of an exact solution of an elastic–plastic problem containing a discontinuous displacement field is given in [8]. Mathematical properties of variational formulations for displacement fields in problems

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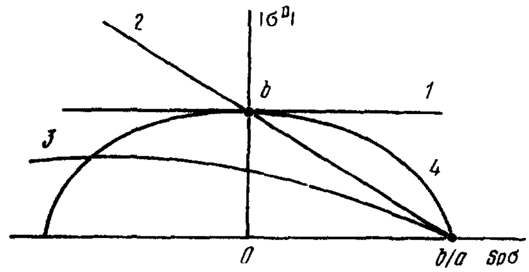

Fig. 1.

with the von Mises criterion were studied in [8–11]. It was shown that the original variational formulation for the displacement field was mathematically ill-posed, because it excluded discontinuous functions. In these papers abstract extensions of the corresponding problems were constructed with the class of admissible functions extended to the space BD of functions of bounded deformation, or to certain of its subsets. As well as complete extension, in many cases it is convenient to use the so-called partially extended variational formulations [7]. Here functionals are defined over a wide class of discontinuous functions and, unlike the case of complete extensions, they have a fairly simple form. The latter turns out to be very important from the applied point of view, because they enable one to construct appropriate numerical methods [12]. The present paper investigates extended variational formulations for problems with yield criteria (0.3)–(0.5). Using them solutions with discontinuous displacement fields are constructed for a range of problems.

The method of constructing these solutions is based on the simultaneous analysis of the displacement problem and its dual stress problem. Since a solution of the latter exists and is unique, it provides the possibility of constructing a lower bound for the displacement functional and of proving that it reaches a minimum only at the discontinuous function.

Moreover, analysis of the extended formulations leads to the conclusion that in problems of the deformational theory of plasticity with criterion (0.1) solutions containing discontinuities along certain lines (surfaces) can occur only when the yield function satisfies the condition

$$F(\mathrm{Sp}\sigma, |\sigma^D|) \geqslant |\sigma^D| + a_* \mathrm{Sp}\sigma - b, \quad b > 0 \tag{0.6}$$

where $a_* = 1/\sqrt{(2)}$ in the plane case and $a_* = 1/\sqrt{(6)}$ in the three-dimensional case.

1. VARIATIONAL FORMULATIONS FOR DEFORMATIONAL PLASTICITY PROBLEMS

The classical formulation of problems of the deformational theory of plasticity consists of determining a stress tensor $\sigma = \sigma(x)$ and a displacement vector $u = u(x)$ satisfying the following system of equations:

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega, \quad i = 1, \dots, n \tag{1.1}$$

$$\sigma_{ij}\nu_j = g_i \quad \text{on} \quad \Gamma_2, \quad u_i = \overset{0}{u}_i \quad \text{on } \Gamma_1, \quad i = 1, \dots, n \tag{1.2}$$

$$\varepsilon_{ij}(u) = {}^1\!/\!_2(u_{i,j} + u_{j,i}), \quad \varepsilon_{ij}(u) = A_{ijkl}\sigma_{kl} + \lambda_{ij}, \quad i, j = 1, \dots, n \tag{1.3}$$

$$F(\sigma) \leqslant 0 \tag{1.4}$$

$$\lambda_{ij}(\tau_{ij} - \sigma_{ij}) \leqslant 0, \quad \forall \tau \in \mathbb{M}_s^{n \times n}; \quad F(\tau) \leqslant 0 \tag{1.5}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ $(n = 2, 3)$ whose boundary Γ is Lipschitz-continuous with

$\Gamma=\Gamma_1\cup\Gamma_2$, $\Gamma_1\cap\Gamma_2=\varnothing$, $\operatorname{mes}\Gamma_1>0$, A_{ijkl} are the components of the elasticity tensor, ν is the unit vector of the outward normal to Γ, f and g are the volume and surface forces u^0 is the displacement specified on Γ_1, $\mathbb{M}_s^{n\times n}$ is the set of symmetric matrices of dimensional $n\times n$, the comma denotes differentiation with respect to the corresponding index, and the summation convention applies to repeated indices from 1 to n. We denote by $L_p(\Omega;\ \mathbb{R}^n)$ and $W_p^l(\Omega;\ \mathbb{R}^n)$ spaces of vector-valued functions defined on Ω whose components are respectively integral up to degree p or belong to the Sobolev class W_p^l.

We define the set of admissible displacement vectors

$$V=\{u(x)\in W_2^1(\Omega;\ \mathbb{R}^n)|u=u^0 \text{ on } \Gamma_1\}$$

and the sets M and Q of stress tensors satisfying, respectively, the equilibrium equations and the given yield condition

$$M=\{\sigma\in\Sigma|\sigma_{ij,j}+f_i=0 \text{ in } \Omega,\ \sigma_{ij}\nu_j=g_i \text{ on } \Gamma_2,\ i=1,\dots,n\}$$

$$Q=\{\sigma\in\Sigma|\sigma(x)\in K \text{ a.e. in } \Omega\},\quad \Sigma=L_2(\Omega;\ \mathbb{M}_s^{n\times n})$$

where

$$K=\{\tau\in\mathbb{M}_s^{n\times n}|F(\tau)\leqslant 0\}$$

We assume that the forces f and g are defined in such a way that a static admissible stress field satisfying the yield condition, i.e. $M\cap Q\neq\varnothing$, exists, and also that

$$f\in L_2(\Omega;\ \mathbb{R}^n),\quad g\in L_\infty(\Gamma_2;\ \mathbb{R}^n),\quad u^0\in W_2^1(\Omega;\ \mathbb{R}^n) \tag{1.6}$$

We define on $Q\times V$ the Lagrangian

$$l(\sigma,u)=\int_\Omega \varepsilon_{ij}(u)\sigma_{ij}dx-G(\sigma)-L(u) \tag{1.7}$$

where

$$L(u)=\int_\Omega f_iu_idx+\int_{\Gamma_2} g_iu_idl,\quad G(\sigma)=\frac{1}{2}\int_\Omega a(\sigma,\sigma)dx$$

and $a(\sigma,\ \sigma)=A_{ijkl}\sigma_{ij}\sigma_{kl}$ is a positive-definite quadratic form. The Lagrangian l generates the following minimax problem: find a pair of functions $(\sigma^*,\ u^*)\in Q\times V$ such that

$$l(\sigma,u^*)\leqslant l(\sigma^*,u^*)\leqslant l(\sigma^*,u)\quad \forall\sigma\in Q,\quad \forall u\in V \tag{1.8}$$

It is easy to show that if a saddle point of the Lagrangian l exists and is attained at sufficiently smooth functions σ^* and u^*, the latter are solutions of the classical formulation (1.1)–(1.5). Conversely, if system (1.1)–(1.5) has a solution, the latter corresponds to the saddle point of l. Hence problem (1.1)–(1.5) can be studied as a minimax problem with respect to the Lagrangian l [11, 13]. The following two problems are associated with it.

Problem P^*: find a tensor $\sigma^*\in Q$ such that

$$\Phi(\sigma^*)=\sup\{\Phi(\sigma)|\sigma\in Q\},\quad \Phi(\sigma)=\inf\{l(\sigma,u)|u\in V\} \tag{1.9}$$

Problem P: find a vector $u^*\in V$ such that

$$J(u^*)=\inf\{J(u)|\ u\in V\},\quad J(u)=\sup\{l(\sigma,u)|\ \sigma\in Q\} \tag{1.10}$$

Calculating the infimum and supremum in (1.9), (1.10) we obtain

$$\Phi(\sigma) = \begin{cases} \int\limits_{\Gamma_1} \sigma_{ij}\nu_i u_j^0 dl - G(\sigma) & \text{if} \quad \sigma \in M \\ -\infty & \text{if} \quad \sigma \notin M \end{cases} \tag{1.11}$$

$$J(u) = \int\limits_{\Omega} H(\varepsilon(u))dx - L(u) \tag{1.12}$$

where $H\colon \mathbb{M}_s^{n\times n} \to \mathbb{R}$ is defined by

$$H(\kappa) = \sup\{\tau_{ij}\kappa_{ij} - G(\tau) \mid \tau \in K\}, \quad \forall \kappa \in M_s^{n\times n}$$

Problems (1.11) and (1.12) are dual with respect to one another, and from convex analysis we know that if solutions σ^* and u^* of problems P^* and P exist, then (σ^*, u^*) is a saddle point of the Lagrangian l on the set $Q \times V$, and conversely, if a pair of functions (σ^*, u^*) satisfying (1.8) exists, then σ^* and u^* are solutions of problems P^* and P, respectively. Here

$$\Phi(\sigma) \leqslant \Phi(\sigma^*) = l(\sigma^*, u^*) = J(u^*) \leqslant J(u), \quad \forall u \in V, \quad \forall \sigma \in Q \tag{1.13}$$

It follows directly from (1.13) that if functions $\sigma^* \in Q$ and $u^* \in Q$ are found such that the equality

$$\Phi(\sigma^*) = J(u^*) \tag{1.14}$$

is satisfied, then σ^* and u^* are solutions of problems P^* and P. We know (see [11, 13]) that problem P^* has a unique solution σ^* if $Q \cap M \neq \emptyset$ and

$$\Phi(\sigma^*) = \sup_{\sigma\in Q}\inf_{u\in V} l(\sigma, u) = \inf_{u\in V}\sup_{\sigma\in Q} l(\sigma, u) = J_* \tag{1.15}$$

However, problem P might not have a solution (an appropriate example for an elastic–plastic problem with the von Mises criterion is given in [8]). This is associated with the fact that a minimizing sequence may converge to a function that has discontinuities along certain lines (surfaces). Since such functions do not belong to the set V and the functional J is not defined on them, it is necessary to use a mathematical extension technique for the given variational problem and to construct an extended problem (see [8–11]). The extended problem P^+ consists of determining an element $u^+ \in V^+$ such that

$$I(u^+) = \inf\{I(\upsilon) \mid \upsilon \in V^+\}, \quad \text{where} \quad V \subset V^+ \tag{1.16}$$

The problem P^+ always has a solution and preserves the exact lower bound of problem P. The problem P^* is also dual to P^+, so that to the stress tensor σ^* there corresponds a variable field $u^+ \in V^+$ with

$$\Phi(\sigma^*) = I(u^+) \tag{1.17}$$

It often turns out to be useful to use the so-called partially extended formulation (see [7]) where the extended functional $\bar{J}$ is defined on a set $\bar{V}$ such that $V \subset \bar{V} \subset V^+$, where $\bar{V}$ contains a selection of certain discontinuous displacement fields. The problem $\bar{P}$ consists of determining a vector-valued function $\bar{u} \in \bar{V}$ which minimizes the functional $\bar{J}$

$$\bar{J}(\bar{u}) = \inf\{\bar{J}(\upsilon) \mid \upsilon \in \bar{V}\} \tag{1.18}$$

Here

$$\bar{J}(u)=J(u) \quad \forall u\in V; \quad \bar{J}(u)=I(u) \quad \forall u\in \bar{V}, \quad \bar{J}(u)\geq J_* \quad \forall u\in \bar{V} \tag{1.19}$$

It follows directly from (1.15)–(1.19) that if a solution of problem P exists, it is also a solution to problem $\bar{P}$ (1.18), and in turn any solution of problem $\bar{P}$ is a solution of problem P^+. Thus if functions $\sigma^*\in Q$ and $\bar{u}\in\bar{V}$ are found such that

$$\Phi(\sigma^*)=\bar{J}(\bar{u}) \tag{1.20}$$

then $\bar{u}$ is a solution of problem P^+, and σ^* is a solution of the dual problem P^*.

We will now restrict our attention to the simplest version of formulation (1.18), when a discontinuity of the displacement field is only permitted at the domain boundary Γ_1. Then [7]

$$\bar{J}(u)=\int_\Omega H(\varepsilon(u))dx-L(u)+\int_{\Gamma_1}\Psi(S(\nu,u-u^0))dl \tag{1.21}$$

$$\bar{V}=W_2^1(\Omega;\mathbb{R}^n), \quad S\in \mathbb{M}_s^{n\times n}, \quad S=\{s_{ij}\}, \quad s_{ij}=\frac{1}{2}(\nu_i\upsilon_j+\nu_j\upsilon_i)$$

and the function Ψ: $\mathbb{M}_s^{n\times n}\to\mathbb{R}$ is computed from

$$\Psi(\kappa)=\sup\{\tau_{ij}\kappa_{ij}|\ \tau\in K\} \quad \forall\kappa\in \mathbb{M}_s^{n\times n} \tag{1.22}$$

Condition (1.20) in which the functional $\bar{J}$ has the form (1.21), can be used to construct exact solutions with displacement discontinuities on the boundary Γ_1.

We will formulate sufficient conditions for the existence of such a solution. To do this we consider an auxiliary problem, which differs from the original one only by the specification of a boundary condition on Γ_1, namely

$$u=w^0 \text{ on } \Gamma_1; \quad w^0\in W_2^1(\Omega;\mathbb{R}^n), \quad w^0\neq u^0 \tag{1.23}$$

The corresponding variational formulation for the displacements has the form

$$\inf\left\{\int_\Omega H(\varepsilon(u))dx-L(u)|u\in V_1\right\} \tag{1.24}$$

$$V_1=\{u\in W_2^1(\Omega;\mathbb{R}^n)\mid u=w^0 \text{ on } \Gamma_1\}$$

Assertion 1. Suppose that a solution of problem (1.24) exists and is attained by the field $u^*$with corresponding stress tensor σ^* (the solution of the corresponding dual problem), and that the following condition is satisfied

$$\int_{\Gamma_1}\{\Psi(S(\nu,w^0-u^0))-\nu_i\sigma^*_{ij}(u^0_j-w^0_j)\}dl=0 \tag{1.25}$$

Then u^* and σ^* are solutions of the problems P^+ and P^*, respectively.

Proof. Since u^* is a solution of problem (1.24) and σ^* is a solution of the corresponding dual problem, then by the duality relations (see (1.14))

$$J(u^*)=-\frac{1}{2}\int_\Omega a(\sigma^*,\sigma^*)dx+\int_{\Gamma_1}\nu_i\sigma^*_{ij}w^0_jdl \tag{1.26}$$

where $\sigma^*\in Q\cap M$. Combining (1.25) and (1.26), we obtain

$$\bar{J}(u^*)=J(u^*)+\int_{\Gamma_1}\Psi(S(\nu,w^0-u^0))dl=-\frac{1}{2}\int_\Omega a(\sigma^*,\sigma^*)dx+\int_{\Gamma_1}\nu_i\sigma^*_{ij}u^0_jdl$$

which shows that condition (1.20) is satisfied and, consequently $u*$ and σ^* are solutions of problems P^+ and P^*.

Thus, if the conditions of Assertion 1 are satisfied, then $u*$ is a solution of the extended problem, which contains a displacement discontinuity of magnitude $w^0 - u^0$ on the boundary Γ_1.

Remark. Let K_1 and K_2 be convex sets corresponding to two different yield conditions F_1 and F_2

$$K_i = \{\tau \in \mathbf{M}_s^{n\times n} \mid F_i(\tau) \leqslant 0\}, \quad i = 1, 2, \quad K_2 \subset K_1 \tag{1.27}$$

We denote the functions P and Ψ corresponding to these sets by H_i and Ψ_i and put

$$Q_i = \{\sigma \in \Sigma \mid \sigma(x) \in K_i \text{ a.e. in } \Omega\}, \quad i = 1, 2$$

We consider two elastic–plastic problems in the domain Ω, differing solely in the choice of yield condition. We assume we have found the solution σ^*, u^* of the $F = F_1$ case, with $\sigma^* \in Q_1$, $u^* \in \overline{V}$ and

$$\begin{gathered}\Phi(\sigma^*) = \chi_1 + \Lambda_1 - L(u^*)\\ \chi_i = \int_\Omega H_i(\varepsilon(u^*))dx; \quad \Lambda_i = \int_{\Gamma_1} \Psi_i(S(\nu, u^* - u^0))dl, \quad i = 1,2\end{gathered} \tag{1.28}$$

If we also have $\sigma^* \in Q_2$ and

$$\Lambda_1 = \Lambda_2 \tag{1.29}$$

then σ^* and u^* are simultaneously also solutions of the second problem (with $F = F_2$).

Indeed, since $\sigma^* \in Q_2$ and $K_2 \subset K_1$, we have

$$\begin{gathered}\Phi(\sigma^*) \leqslant \sup\{\Phi(\sigma) \mid \sigma \in Q_2\} \leqslant \sup\{\Phi(\sigma) \mid \sigma \in Q_1\} = \Phi(\sigma^*)\\ H_2(\kappa) \leqslant H_1(\kappa) \quad \forall \kappa \in \mathbf{M}_s^{n\times n} \Rightarrow \chi_2 \leqslant \chi_1\end{gathered} \tag{1.30}$$

Using (1.28)–(1.30), we obtain

$$\sup\{\Phi(\sigma) \mid \sigma \in Q_2\} = \Phi(\sigma^*) = \chi_1 + \Lambda_1 - L(u^*) \geqslant \chi_2 + \Lambda_2 - L(u^*) \geqslant \Phi(\sigma) \quad \forall \sigma \in Q_2$$

Here we put $\sigma = \sigma^*$ and conclude that condition (1.20) is satisfied in the second $(F = F_2)$ problem, so that σ^* and u^* are solutions.

2. EXTENDED FORMULATIONS FOR PROBLEMS WITH YIELD CONDITIONS (0.3)–(0.5)

For an isotropic medium the quadratic form $a(\tau, \tau)$ can be written in the form

$$a(\tau, \tau) = \frac{1}{2\mu} |\tau^D|^2 + \frac{1}{n^2 K_0} (\mathrm{Sp}\tau)^2$$

where μ and K_0 are constants of elasticity for the material. In the case under consideration the yield conditions also depend solely on the deviator and trace of the tensor, and so the functions H and Ψ defined by (1.12) and (1.22) can be written as functions of two parameters $s = \mathrm{Sp}\varepsilon$ and $t = |\varepsilon^D|$. For the von Mises yield condition the expressions for H and Ψ are known to be [13]

$$H(t,s)=\frac{K_0}{2}s^2+h_0(t);\quad h_0(t)=\begin{cases}\mu t^2, & t\leqslant k_*/\sqrt{2}\mu\\ k_*(\sqrt{2}t-k_*/2\mu), & t>k_*/\sqrt{2}\mu\end{cases}$$

$$\Psi(\kappa)=\begin{cases}\sqrt{2}k_*|\kappa^D|, & \mathrm{Sp}\,\kappa=0\\ +\infty, & \mathrm{Sp}\,\kappa\neq 0\end{cases}$$

We will consider other cases.

The Coulomb–Mohr yield condition. Computing the supremum over the set K, corresponding to condition (0.3), we obtain

$$H(t,s)=\begin{cases}\mu t^2+\dfrac{K_0}{2}s^2, & \text{if}\quad t+\dfrac{as}{nq}\leqslant\dfrac{b}{2\mu}\\ h_1(t,s)+\dfrac{\mu h_2^2(t,s)}{a^2+q} & \text{if}\quad t+\dfrac{as}{nq}>\dfrac{b}{2\mu}\ \text{and}\ h_2(t,s)\geqslant 0\\ h_1(t,s), & \text{if}\quad h_2(t,s)<0\end{cases}\tag{2.1}$$

where

$$q=\frac{2\mu}{n^2K_0},\quad h_1(t,s)=\frac{b}{a}\left(\frac{s}{n}-\frac{qb}{4\mu a}\right),\quad h_2(t,s)=at-\frac{s}{n}+\frac{qb}{2\mu a}$$

$$\Psi(\kappa)=\begin{cases}\dfrac{b\,\mathrm{Sp}\,\kappa}{na}, & \text{if}\quad |\kappa^D|\leqslant\dfrac{1}{na}\mathrm{Sp}\,\kappa\\ +\infty, & \text{otherwise}\end{cases}\qquad\forall\kappa\in\mathbf{M}_s^{n\times n}\tag{2.2}$$

Since

$$|S^D(\nu,\upsilon)|=\left(\frac{n-1}{n}|\upsilon_\nu|^2+\frac{1}{2}|\upsilon_\tau|^2\right)^{1/2},\quad \upsilon_\nu=\nu_i\upsilon_i=\mathrm{Sp}(S(\nu,\upsilon))\upsilon_\tau=\upsilon-\upsilon_\nu\nu$$

then $\Psi(S(\nu,\ \upsilon))$ can be finite only when

$$a\leqslant a_*=1/(n^2-n)^{1/2}\tag{2.3}$$

When $a>a_*$, $\psi(S(\nu,\ \upsilon)=+\infty$ for any non-zero vector υ. Thus a solution containing a discontinuity (which corresponds to $\upsilon\neq 0$) along some line or surface can exist only if the constant a in condition (0.3) satisfies condition (2.3).

Yield condition (0.4). Explicit expressions for H and Ψ cannot be constructed for arbitrary convex functions h. However, one can establish some important properties of these functions. We introduce the notation

$$K_{ab}=\{\tau\in\mathbf{M}_s^{n\times n}\,|\,|\tau^D|+a\mathrm{Sp}\tau\leqslant b\},\quad \Psi_{ab}(\kappa)=\sup\{\tau_{ij}\kappa_{ij}|\,\tau\in K_{ab}\}$$

As a direct consequence of the definition of Ψ we have the following assertion.

Assertion 2. If $K_{a_1b_1}\subset K\subset K_{a_2b_2}$, then

$$\Psi_{a_1b_1}(\kappa)\leqslant\Psi(\kappa)\leqslant\Psi_{a_2b_2}(\kappa)\qquad\forall\kappa\in\mathbf{M}_s^{n\times n}$$

Corollary. Suppose that the set K corresponding to condition (0.1) contains the set K_{ab} where $a\geqslant a_*$ and $b>0$ (or, equivalently, that inequality (0.6) is satisfied). Then $\Psi(S(\nu,$

$\upsilon)) = +\infty$ for any $\upsilon \neq 0$, and, consequently, discontinuous solutions are impossible.

The value of the function Ψ can be calculated in two different cases, corresponding to normal and tangential discontinuities.

Assertion 3. 1. Suppose that $K \subset K_{ab}$ when $a \leqslant a$, $b > 0$ and

$$\tau^0 = \frac{b}{na} 1 \in K$$

where $1 \in M_s^{n \times n}$ is the unit tensor, then

$$\Psi(S(\nu, \upsilon)) = \upsilon_\nu b/(na), \quad \text{if} \quad |\upsilon_\tau| = 0, \quad \upsilon_\nu > 0 \tag{2.4}$$

2. Suppose that the function H in (0.4) satisfies the condition

$$inf\{h(t)|\, t \in \mathbf{R}\} = c > -\infty \tag{2.5}$$

then

$$\Psi(S(\nu, \upsilon)) = (b-c)|\upsilon_\tau|/\sqrt{2}\,, \quad \text{if} \quad \upsilon_\nu = 0. \tag{2.6}$$

Proof. 1. In the case $|\upsilon_\tau| = 0$, $\upsilon_\nu > 0$ the condition $|S^D(\nu, \upsilon)| \leqslant \mathrm{Sp} S(\nu, \upsilon)/(na)$ is obviously satisfied when $a < a$, and using (2.2) we obtain

$$\Psi(S(\nu, \upsilon)) = \sup\{\tau_{ij}s_{ij}|\, \tau \in K\} \leqslant \sup\{\tau_{ij}s_{ij}|\, \tau \in K_{ab}\} = b/na\upsilon_\nu$$

On the other hand

$$\sup\{\tau_{ij}s_{ij}|\, \tau \in K\} \geqslant \overset{0}{\tau_{ij}}s_{ij} = b/na\upsilon_\nu$$

and we thus obtain (2.4).

2. To prove the second part of the assertion we note that if $\upsilon_\nu = 0$, then $|S^D(\nu, \upsilon)| = |\upsilon_\tau| \sqrt{(s)}$

$$\Psi(S(\nu, \upsilon)) = \sup\{\tau_{ij}s_{ij}|\, \tau \in K\} = D|S^D(\nu, \upsilon)\}, \; D = \sup\{|\tau^D|\,|\, \tau \in K\}$$

Since $|\tau^D| \leqslant b - h(\mathrm{Sp}\tau)$, it follows from (2.5) that $D = b - c$ and that (2.6) holds.

Yield condition (0.5). Here the expressions for Ψ and H have the form

$$\Psi(\kappa) = \frac{b}{na}\sqrt{|\mathrm{Sp}\,\kappa|^2 + n^2a^2|\kappa^D|^2}, \quad \kappa \in \mathbf{M}_s^{n \times n} \tag{2.7}$$

$$H(t,s) = \begin{cases} \mu t^2 + \dfrac{K_0}{2}s^2, & \text{if} \quad t^2 + \dfrac{a^2s^2}{q^2n^2} \leqslant \dfrac{b^2}{4\mu^2} \\ \lambda_*(|s|/n - q\lambda_*/4\mu) + \delta_* b(t - \delta_* b/4\mu), & \text{if} \quad t^2 + \dfrac{a^2s^2}{q^2n^2} > \dfrac{b^2}{4\mu^2} \end{cases} \tag{2.8}$$

where $\lambda_* = ba^{-1}\sqrt{(1-\delta_*^2)}$, and $\delta_* \in [0, 1]$ is defined to be the root of the equation

$$|s|\delta/(an\sqrt{1-\delta^2}) = t - \delta b(A/a^2 - 1)/(2\mu)$$

if $A = a^2$ (which will, for example, be the case for an incompressible material in a plane stress state), expression (2.8) is simplified considerably

$$H(t,s)=\begin{cases}\mu\Theta^2 & \text{if } |\Theta|\leqslant b/2\mu\\ b(|\Theta|-b/4\mu) & \text{if } |\Theta|>b/2\mu\end{cases}$$

where $\Theta^2=s^2+t^2/(a^2n^2)$. We note that here, according to (2.7), discontinuities are possible in the normal as well as the tangential field components. However, similar solutions cannot be treated as a real discontinuity of the medium. In plane stress state problems, solutions containing such discontinuities are associated with the formation of a neck [4], while in three-dimensional problems they can be considered as a mathematical description of localization processes in plastic deformation [14], which are often accompanied by material decohesion and dilation phenomena [15, 16].

3. SOME EXACT SOLUTIONS

We will use the results obtained in the first two sections to obtain exact solutions of the problems of a twisted thick-walled cylinder and the deformed spherical layer, and to show that for a broad collection of yield conditions the solution obtained has a discontinuous displacement field.

3.1. Suppose $\Omega=\{(r,\Theta,z)\,|\,r_1\leqslant r\leqslant r_2,\ 0\leqslant\Theta<2\pi,\ -d\leqslant z\leqslant d\}$, $u=(u_r,\ u_\Theta,\ u_z)$, where (r,Θ,z) are cylindrical coordinates. We consider a problem with yield criterion (0.4), where

$$h(t)\geqslant 0,\quad h(0)=0 \tag{3.1}$$

and boundary conditions

$$u=(0,0,0)\text{ when } r=r_1,\quad u=(0,U,0)\text{ when } r=r_2,\quad U=\text{const} \tag{3.2}$$

$$\sigma_{zz}=\sigma_{rz}=\sigma_{\theta z}=0\text{ when } z=\pm d \tag{3.3}$$

where $U\geqslant U_*=\beta b/2\sqrt{2}\mu r_2$, $\beta=r_2^2-r_1^2$, and assume that the external forces f and g are zero. To construct an exact solution of this problem we use Assertion 1, where the function w^0 satisfies the conditions

$$w^0=\overset{0}{w}_1=(0,\upsilon,0)\text{ when } r=r_1,\quad w^0=\overset{0}{w}_2=(0,U,0)\text{ when } r=r_2 \tag{3.4}$$

$$\upsilon=r_1\overset{-1}{r_2}[U-U_*]>0$$

It is easy to verify that in this case the solution of the problem exists (1.24) and is determined as follows:

$$\overset{*}{u}_r=\overset{*}{u}_z=0,\quad \overset{*}{u}_\theta=C_1r+C_2r^{-1} \tag{3.5}$$

$$\overset{*}{\sigma}_{zz}=\overset{*}{\sigma}_{rz}=\overset{*}{\sigma}_{\theta z}=\overset{*}{\sigma}_{rr}=\overset{*}{\sigma}_{\theta\theta}=0,\quad \overset{*}{\sigma}_{r\theta}=-2\mu C_2r^{-2} \tag{3.6}$$

$$C_1=\beta^{-1}(Ur_2-\upsilon r_1),\quad \overset{-1}{C_2}=\beta^{-1}r_1r_2(\upsilon r_2-Ur_1)$$

We note that $\sigma^*\in K\cap M$. Condition (1.25), using the axisymmetry of u^* and σ^*, can be written in the form

$$\Psi(S(\nu,\overset{0}{w}_1))=\overset{*}{\sigma}_{r\theta}(r_1)\upsilon,\quad \nu=(-1,0,0) \tag{3.7}$$

With the help of Assertion 3 and using (3.1) we obtain

$$\Psi(S(v, w_1^0)) = b|\upsilon|/\sqrt{2} \tag{3.8}$$

Since $\upsilon > 0$ and $C_2 = -r_1^2 b/2\sqrt{2}\mu$, it follows from (3.6)–(3.8) that (1.25) is satisfied. Thus u^* and σ^* defined by (3.5) and (3.6) are solutions of the problem with boundary conditions (3.2) and (3.3) and the magnitude of the displacement discontinuity for $r = r_1$ is equal to υ. Note that $\varepsilon(u^*)$ is the unique tensor satisfying relations (1.3)–(1.5) for $\sigma = \sigma^*$, so that this problem has no other solutions.

We thus conclude that for any yield law (0.4) satisfying condition (3.1) (see Fig. 2) the problem of a twisted thick-walled cylinder with conditions (3.2)–(3.4) has a slip-type discontinuity at $r = r_1$.

3.2. Let $\Omega = \{(r, \Theta, \varphi) \mid r_1 \leqslant r \leqslant r_2,\ 0 \leqslant \Theta \leqslant \pi,\ 0 \leqslant \varphi \leqslant 2\pi\}$, $u = (u_r, u_\Theta, u_\varphi)$, where (r, Θ, φ) are spherical coordinates.

We consider the elastic–plastic problem in a domain Ω assuming that the material obeys the yield condition (0.3) and that there are no volume forces f. We specify the following boundary conditions

$$u = (0, 0, 0) \text{ when } r = r_1, \quad u = (U, 0, 0) \text{ when } r = r_2 \tag{3.9}$$

where $U = Cr_2$, $C = b/9aK_0$, $a \leqslant a^*$.

Using (2.1) it is easy to show that condition (1.20) is satisfied when u^* and σ^* have the form

$$u^* = (\overset{*}{u}_r, 0, 0), \quad \overset{*}{u}_r = Cr, \quad \overset{*}{\sigma}_r = \overset{*}{\sigma}_\theta = \overset{*}{\sigma}_\varphi = b/3a \tag{3.10}$$

$$\overset{*}{\sigma}_{r\varphi} = \overset{*}{\sigma}_{r\theta} = \overset{*}{\sigma}_{\varphi\theta} = 0$$

Here

$$\bar{J}(u^*) = \Phi(\sigma^*) = \frac{2\pi b}{3a} C(r_2^3 + r_1^3), \quad \sigma^* \in Q \cap M \tag{3.11}$$

According to (1.20) this means that σ^* and u^* are solutions of the variational problems P^* and P^+. We note that $u_r^* = Cr_1$ when $r = r_1$ and, consequently, the infimum of the extended functional is attained at a discontinuous function.

We now consider the same problem, except that the yield condition is criterion (0.4) rather than (0.3), with the function h specified by

$$h(t) \geqslant a_* t, \quad h(b/a) = b \tag{3.12}$$

The corresponding curves $|\sigma^D| = b - h(\mathrm{Sp}\sigma)$ are shown in Fig. 3 (curves 1–3). We will show that u^* and σ^* defined by (3.10) are also solutions of this problem. To this end we use the remark in Assertion 1 and put

$$F_1(\sigma) = |\sigma^D| + a_* \mathrm{Sp}\sigma - b, \quad F_2(\sigma) = |\sigma^D| + h(\mathrm{Sp}\sigma) - b$$

Since $|\sigma^{*D}| + h(\mathrm{Sp}\sigma^*) = h(b/a) = b$ we have $\sigma^* \in Q_2$. Moreover, from (2.2) and the first part of Assertion 1 it follows that (1.29) is satisfied. Thus u^* and σ^* satisfy (3.11), and the infimum of $\bar{J}(u)$ is attained at a discontinuous function.

In the case when $a = a_* = 1/\sqrt{6}$ one can construct an exact solution of the problem of a deformed spherical layer with yield condition (0.3) and boundary conditions (3.9) when $U \leqslant U_2 = 2br_2/3\sqrt{(6)}K_0$. If $U < U_1 = 2\beta b/\sqrt{(6)}r_2^2(3K_0 + 4\mu)$, $\beta = r_2^3 - r_1^3$, the solution is elastic

$$\overset{*}{u}_\varphi = \overset{*}{u}_\theta = 0, \quad \overset{*}{u}_r = \beta^{-1} U r_2^2 r[1 - r_1^3/r^3]$$

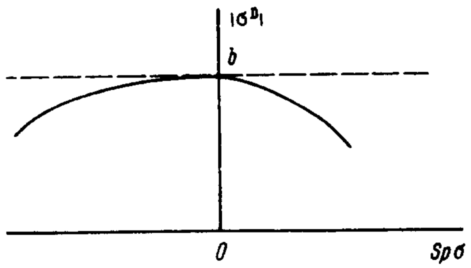

Fig. 2.

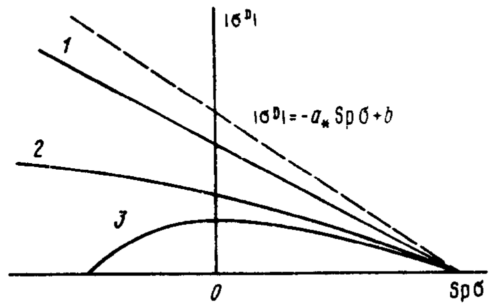

Fig. 3.

In order to construct a solution for $U_1 < U < U_2$ we use Assertion 1, and choose the function w^0 in (1.24) so that

$$w^0 = \overset{0}{w}_1 \text{ when } r = r_1, \quad w^0 = \overset{0}{w}_2 \text{ when } r = r_2$$

$$\overset{0}{w}_1 = (\upsilon, 0, 0), \quad \overset{0}{w}_2 = (U, 0, 0)$$

where the constant υ is defined by the equation

$$\frac{v}{r_1}\left(2r_2^3 + r_1^3\frac{3K_0}{2\mu}\right) = \frac{3K_0+4\mu}{2\mu}r_2^2U - \frac{b\beta}{\sqrt{6}\mu} \tag{3.13}$$

We note that $\upsilon > 0$ when $U_1 < U < U_2$. In this case problem (1.24) has the following solution

$$\overset{*}{u}_\varphi = \overset{*}{u}_\theta = 0, \quad \overset{*}{u}_r = C_1 r + C_2 r^{-2}, \quad \overset{*}{\sigma}_{r\theta} = \overset{*}{\sigma}_{r\varphi} = \overset{*}{\sigma}_{\theta\varphi} = 0 \tag{3.14}$$

$$\overset{*}{\sigma}_r = 3K_0C_1 - \frac{4\mu}{r^3}C_2, \quad \overset{*}{\sigma}_\theta = \overset{*}{\sigma}_\varphi = 3K_0C_1 + \frac{2\mu}{r^3}C_2$$

where

$$C_1 = (Ur_2^2 - \upsilon r_1^2)\beta^{-1}, \quad C_2 = r_2^2 r_1^2(\upsilon r_2 - Ur_1)\beta^{-1} \tag{3.15}$$

Taking (2.2) and (3.14) into account, condition (1.25) can be written in the form

$$3K_0C_1/2\mu - 2C_2/r_1^3 = b/\sqrt{6}\,\mu$$

Using (3.13) and (3.15), it is easy to verify that this equality is satisfied. Thus u^* and σ^* defined by formulae (3.14) and (3.15) are solutions of the problem with boundary condition (3.9). For $U_1 < U < U_2$ this solution has a discontinuity of magnitude υ at $r = r_1$. We note that since

$$F(\sigma^*) = \frac{2\sqrt{6}\mu}{r^3}|C_2| + \frac{3\sqrt{6}K_0}{2}C_1 - b < 0 \qquad \forall r \in (r_1, r_2]$$

then, according to (1.5), $\lambda_{ij} \equiv 0$ and, consequently, $\varepsilon(u^*)$ is the unique tensor which together with σ satisfies relations (1.4) and (1.5).

REFERENCES

1. DRUCKER D. C. and PRAGER W., Soil mechanics and plastic analysis or limit design. *Quart. Appl. Math.* **10**, 2, 157–165, 1952.
2. NADAI A., *Plasticity and the Fracture of Solids*. IIL, Moscow, 1954.
3. GREEN R. J., A plasticity theory for porous solids. *Int. J. Mech. Sci.* **14**, 4, 215–224, 1972.
4. KACHANOV L. M., *Foundations of the Theory of Plasticity*. Nauka, Moscow, 1969.
5. IVLEV D. D., *The Theory of Ideal Plasticity*. Nauka, Moscow, 1966.
6. DRUGAN W. J. and RICE J. R., Restrictions on quasi-statically moving surfaces of strong discontinuity in elastic–plastic solids. In *Mechanics of Material Behaviour* (Edited by G. J. Dvorak and R. T. Shield), pp. 59–73. Elsevier, Amsterdam.
7. REPIN S. I., Variational formulations for discontinuous displacement fields in problems of the deformation theory of plasticity without hardening. *Prikl. Mat. Mekh.* **55**, 6, 1026–1034, 1991.
8. SEREGIN G. A., On the correctness of variational problems in the mechanics of ideally elastic–plastic media. *Dokl. Akad. Nauk SSSR* **276**, 1, 71–75, 1984.
9. SEREGIN G. A., On formulations of problems in the theory of ideally elastic–plastic bodies. *Prikl. Mat. Mekh.* **49**, 5, 849–859, 1985.
10. ANZELLOTI G. and GIAQUINTA M., Existence of the displacement field for an elastic–plastic body subject to Hencky's law and the von Mises yield condition. *Manuscripta Math.* **32**, 1–2, 101–136, 1980.
11. TEMAM R. and STRANG G., Duality and relaxation in the variational problems of plasticity. *J. Mechanique* **19**, 3, 493–527, 1980.
12. REPIN S. I., A variational-difference method for solving problems of ideal plasticity taking into account the possible appearance of discontinuities. *Zh. Vychisl. Mat. Fiz.* **28**, 3, 449–453, 1988.
13. DUVAT G. and LIONS J.-L., *Inequalities in Mechanics and Physics*. Springer, Berlin, 1976.
14. RICE, J. R., Localization of plastic deformation. In *Theoretical and Applied Mechanics: Proceedings of the 14th Intern. Cong. IUTAM* (Edited by V. T. Koiter), pp. 439–471. Mir, Moscow, 1979.
15. SZUWALSKI K. and ZYCZKOWSKI M., On the phenomenon of decohesion in perfect plasticity. *Int. J. Solids and Structures* **9**, 1, 85–98, 1973.
16. RUDNICKI J. W. and RICE J. R., Conditions for the localization of deformation in pressure-sensitive dilatant materials. *J. Mech. Phys. Solids* **23**, 6, 371–394, 1975.

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